1801. The intersections are at $kx = x^2$, so $x = 0$ and $x = k$. Assuming positive k, the line is above the parabola, so the area is given by

$$\begin{aligned}&\int_0^k kx - x^2\,dx\\ \equiv\ &\left[\tfrac{1}{2}kx^2 - \tfrac{1}{3}x^3\right]_0^k\\ \equiv\ &\left(\tfrac{1}{2}k^3 - \tfrac{1}{3}k^3\right) - (0)\\ \equiv\ &\tfrac{1}{6}k^3.\end{aligned}$$

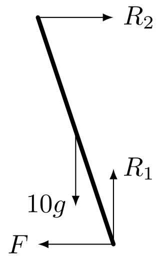
If k is negative, the calculation is the same, but its result $\frac{1}{6}k^3$ is negative. Hence, to find the area, which is always positive, we need to apply the modulus function, giving $\left|\frac{1}{6}k^3\right|$, as required.

1802. Using the binomial expansion, the terms are

$$x^{\frac{3}{2}} \pm 3x(x+1)^{\frac{1}{2}} + 3x^{\frac{1}{2}}(x+1) \pm (x+1)^{\frac{3}{2}}.$$

Upon adding, the second and fourth terms cancel, leaving $2x^{\frac{3}{2}} + 6x^{\frac{1}{2}}(x+1)$. This has a common factor of $2\sqrt{x}$, which gives $2\sqrt{x}(4x+3)$.

1803. (a) The force diagram for the sheet is

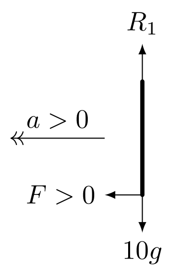


Taking moments around the base,

$$R_2 = l\sin 75° - 10g \cdot \tfrac{1}{2}l\cos 75° = 0,$$

where l is the slant height of the plywood. This gives $R_2 = 49(2-\sqrt{3}) = 13.1$ N (3sf).

(b) The van is accelerating to the left. So, since the floor is rough enough to rule out sliding, the base of the plywood must accelerate to the left also. Hence, there must be a frictional force, magnitude $F > 0$, acting leftwards on the base of the sheet. And, since the sheet is vertical, neither the reaction from the floor nor the weight can have a turning effect.



So, there is a resultant moment clockwise around the base. This will cause the plywood to topple towards the opposite wall.

1804. Any non-constant GP, together with any linear function involving addition of a constant, provides a counterexample. Using $1, 2, 4, 8, 16, ...$ as the GP, consider $f(x) = x + 1$. The sequence $2, 3, 5, 9, 17, ...$ is not a GP, as $3/2 \neq 5/3$.

1805. The derivative is a standard result, which comes directly from applying the chain rule to $(\cos x)^{-1}$: $\frac{du}{dx} = \sec x \tan x$. The relevant exact values are $\cos(\frac{\pi}{6}) = \sqrt{3}/2$ and $\tan(\frac{\pi}{6}) = \sqrt{3}/3$. So,

$$\left.\frac{du}{dx}\right|_{x=\frac{\pi}{6}} = \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{3} = \frac{2}{3}.$$

1806. The regions are the interiors of two circles: one with radius 2, centred on $(0,-4)$, the other with radius $\sqrt{6}$, centred on $(2,0)$. The distance between the centres is $\sqrt{4^2+2^2} = \sqrt{20}$. The sum of the two radii is $2+\sqrt{6} < \sqrt{20}$. The circles, therefore, do not intersect. Hence, there are no points which are simultaneously in both interiors.

1807. We make the square root the subject, enabling us to square both sides and get rid of it:

$$\begin{aligned}x &= \frac{y - \sqrt{y^2+16y}}{8}\\ \Longrightarrow\ \sqrt{y^2+16y} &= y - 8x\\ \Longrightarrow\ y^2 + 16y &= y^2 - 16xy + 64x^2\\ \Longrightarrow\ y &= -xy + 4x^2.\end{aligned}$$

Gathering the terms in y and factorising,

$$\begin{aligned}&y(x+1) = 4x^2\\ \Longrightarrow\ &y = \frac{4x^2}{x+1}.\end{aligned}$$

1808. The exponential function, which is the inverse of the natural logarithm function, cannot be applied to the individual terms, as this student has done. Exponentiation doesn't distribute over addition. Instead, log (or index) rules must be used:

$$\begin{aligned}&\ln x + \ln(x-1) = \ln 6\\ \Longrightarrow\ &\ln(x^2 - x) = \ln 6\\ \Longrightarrow\ &x^2 - x = 6\\ \Longrightarrow\ &x = -2, 3.\end{aligned}$$

Since the natural logarithm function is undefined for $x = -2$, the solution is $x = 3$.

1809. This is false. Solving $y^3 - y = 0$ gives $y = -1, 0, 1$, each of which is a single root. Hence, the y axis intersects the curve three times, but is not tangent to it.

1810. The mean is a measure of central tendency, and is therefore affected by both scaling and translation; it is transformed in the obvious way.

The variance, however, is a squared measure of spread, and is unaffected by translation; it is only scaled by a^2.

1811. The implication holds in both directions. The first statement says that the positive fourth root of x is 2: there is only one x satisfying this, which is 16. And, in the other direction, the positive fourth root of 16 is 2. So, $\sqrt[4]{x} = 2 \iff x = 16$.

1812. Factorising the quadratic in $\tan 2x$,

$$\begin{aligned} &\tan^2 2x - \sqrt{3}\tan 2x = 0 \\ \Longrightarrow\ &\tan 2x(\tan 2x - \sqrt{3}) = 0 \\ \Longrightarrow\ &\tan 2x = 0, \sqrt{3}. \end{aligned}$$

Firstly, $\tan 2x = 0$ gives $2x = 0, \pi, 2\pi$. Secondly, $\tan 2x = \sqrt{3}$ gives $2x = \pi/3, 4\pi/3$. Dividing by two, the solution set is $\{0, \pi/6, \pi/2, 2\pi/3, \pi\}$.

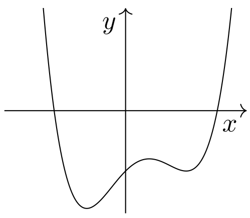
1813. (a) The derivative is

$$\frac{dy}{dx} = 4x^3 - 3x^2 - 2x + 1.$$

For SPs, we set this to zero and solve. Since $x = 1$ is a stationary point, we know, using the factor theorem, that $(x - 1)$ is a root:

$$\begin{aligned} &4x^3 - 3x^2 - 2x + 1 = 0 \\ \Longrightarrow\ &(x-1)(4x^2 + x - 1) = 0 \\ \Longrightarrow\ &x = 1, \tfrac{-1\pm\sqrt{17}}{8}. \end{aligned}$$

(b) Since the equation is a positive quartic, the three stationary points must be two minima at $x = \frac{-1-\sqrt{17}}{8}$ and $x = 1$ and a maximum in between then at $x = \frac{-1+\sqrt{17}}{8}$. So, the graph is



1814. Since chairperson and secretary are different roles, we can simply pick them one after the other; we don't need to worry about the order. There are 10 choices for chairperson, and then 9 for secretary. This gives 90 ways in total.

1815. We need to show that $(qx - 1)$ is never a factor of $x^2 + qx + 1$. By the factor theorem, we need to show that $x = 1/q$ is not a root of $x^2 + qx + 1 = 0$. Subbing in, the LHS is $1/q^2 + 2$. But this is never zero, as the fraction $1/q^2$ is positive. So, $x^2 + qx + 1$ leaves a remainder when divided by $(qx - 1)$. $\square$

1816. (a) Travelling on the base, then the shortest length is directly from A to C, length $\sqrt{2}$.

(b) If travel on the base is not possible, then the shortest length is via the midpoint of BX (or equivalently DX). Using standard trig values for the sloped faces, which are equilateral, this distance is $2 \times \sqrt{3}/2 = \sqrt{3}$.

1817. The first statement is true, as the output of e^{2x-3} can never be zero. The second statement is false, as the output of a logarithm can be zero. The counterexample is $x = 2$.

— Nota Bene —

It is the *inputs* of a logarithm that can't be zero, which isn't relevant to this question.

1818. (a) Both logarithms must be well defined, with positive inputs, so the domain is $(0, 4)$.

(b) Differentiating twice,

$$\begin{aligned} &\mathrm{f}'(x) = -\frac{1}{4-x} - \frac{1}{x} \\ \Longrightarrow\ &\mathrm{f}''(x) = -\frac{1}{(4-x)^2} + \frac{1}{x^2}. \end{aligned}$$

At a point of inflection, $\mathrm{f}''(x) = 0$, so

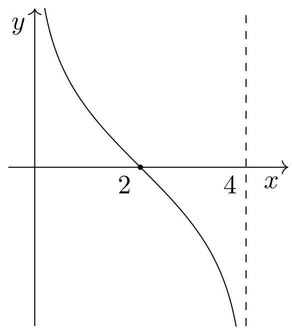
$$\begin{aligned} &-\frac{1}{(4-x)^2} + \frac{1}{x^2} = 0 \\ \Longrightarrow\ &-x^2 + (4-x)^2 = 0 \\ \Longrightarrow\ &16 - 8x = 0 \\ \Longrightarrow\ &x = 2. \end{aligned}$$

Substituting this into the original equation gives $y = 0$. We can then establish that $\mathrm{f}''(x)$ changes sign at $x = 2$ by simplifying to

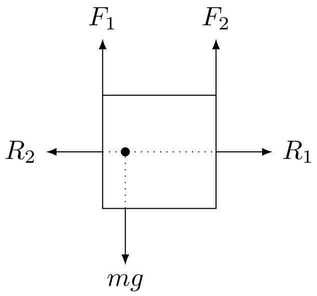
$$\mathrm{f}''(x) = \frac{-8(x-2)}{(4-x)^2x^2}.$$

The numerator has a single root at $x = 2$, so the fraction changes sign there. Hence, $(2, 0)$ is a point of inflection.

(c) The curve has vertical asymptotes at $x = 0$ and $x = 4$. At the former, $y \to \infty$; at the latter, $y \to -\infty$. So, the curve is



1819. (a) The force diagram for the rock climber is



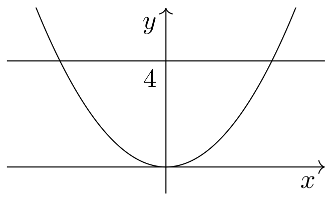
Since the reaction forces are the only forces acting horizontally, $R_1 = R_2$.

(b) Around the point marked above, the reaction forces and weight have no moment. Hence, the frictional forces must be in the ratio $4 : 1$, with F_1 larger than F_2. The centre of mass of the climber is closer to the wall away from which he or she is facing, because his or her torso is there; hence, a greater frictional force is required there.

(c) Oriented as above, the maximum frictional forces are $\frac{1}{3}R$ at the back and $\frac{1}{5}R$ at the feet. Since the frictional force at the back is 4 times larger, that is where slipping will first occur. Vertical equilibrium gives

$$\begin{aligned} & F_{\text{max}} + \tfrac{1}{4}F_{\text{max}} - mg = 0 \\ \implies & \tfrac{1}{3}R + \tfrac{1}{12}R - mg = 0 \\ \implies & R = \tfrac{12}{5}mg. \end{aligned}$$

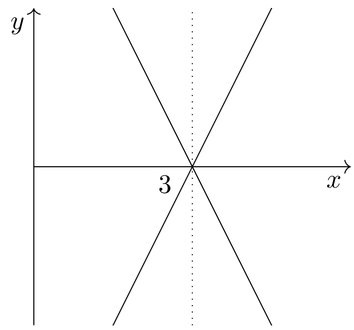
1820. The equation $(y - x^2)(y - 4) = 0$ is satisfied when either $y = x^2$ or $y = 4$. These two divide the plane up into five distinct regions:



The region enclosed by the two curves satisfies the inequality, because $y - x^2$ and $y - 4$ have opposite signs. Furthermore, since crossing any line negates the sign of one of the factors, the top-left and top-right regions must also satisfy the inequality. This gives three distinct regions.

1821. Differentiating a polynomial reduces its degree by one. So, if f'' is quadratic, then f must be quartic, of the form $\mathrm{f}(x) = ax^4 + bx^3 + cx^2 + dx + e$, where $a, b, c, d, e \in \mathbb{R}$ and $a \neq 0$.

1822. The given lines meet at $(3, 0)$, and, since they have gradients ± 2, are symmetrical in $x = 3$ and $y = 0$.



These are the angle bisectors of lines A_1 and A_2, and so are the lines we require.

1823. (a) Substituting the exact value for $\cos\frac{\pi}{6}$,

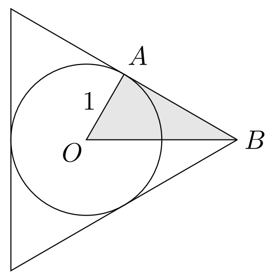
$$\begin{aligned} & \tfrac{\sqrt{3}}{2} \approx 1 - \frac{\pi^2}{72} \\ \implies & \pi^2 \approx 36\left(2 - \sqrt{3}\right) \\ \therefore \quad & \pi \approx 6\sqrt{2 - \sqrt{3}}. \end{aligned}$$

(b) The percentage error is

$$\frac{\pi - 6\sqrt{2 - \sqrt{3}}}{\pi} = 1.14\% \text{ (2dp)}.$$

Even though $\frac{\pi}{6}$ radians is not a particularly small angle, the approximation still holds well, with only a 1% error.

1824. Setting the radius to 1, the scenario is



Using the exact value of $\tan 60°$, $|AB| = \sqrt{3}$. So, the area of triangle AOB is $\sqrt{3}/2$. The equilateral triangle consists of 6 such triangles, so it has area $3\sqrt{3}$. Hence, since the area of the circle is π, the ratio of areas is $3\sqrt{3} : \pi$.

1825. (a) The possibility space consists of $6 \times 12 = 72$ outcomes. There are two outcomes which give $S = 3$, namely $(1, 2)$ and $(2, 1)$. Hence,

$$\mathbb{P}(S = 3) = \tfrac{2}{72} = \tfrac{1}{36}.$$

(b) Since $S = 3$, the restricted possibility space consists of two outcomes. Of these, one has the dodecahedron showing a 2. Hence,

$$\mathbb{P}(\text{dodecahedron shows } 2 \mid S = 3) = \tfrac{1}{2}.$$

There are 10 trials, so the expected number is $np = 10 \times \frac{1}{2} = 5$.

1826. The boundary ellipse has equation $16x^2+y^2=15$. Solving for any intersections, we can eliminate y^2 by adding. This gives $17x^2=16$, so $x^2={}^{16}/_{17}$. Substituting back in gives $y^2=-{}^1/_{17}$, which has no roots. Hence, the hyperbola doesn't intersect the boundary ellipse. Furthermore, the hyperbola is outside the ellipse, as can be verified by e.g. $\big(\pm\sqrt{100},\pm\sqrt{99}\big)$. This gives the required result.

1827. (a) Yes, the graphs intersect at $\big(0,\mathrm{p}(0)\big)$.
   (b) No, a counterexample is $\mathrm{p}(x)=x^2+1$.
   (c) No, with the same counterexample.

1828. Let us assume that the shapes aren't degenerate, i.e. that no three vertices are collinear.
   (a) The lower bound is 0 (degenerate), which isn't attainable; the upper bound is ab (rectangle), which is attainable. The set of possible areas is $(0,ab]$.
   (b) The lower bound is 0 (degenerate), which isn't attainable; the upper bound is $2\times\frac{1}{2}ab$ (right-angled kite), which is attainable. Again, the set of possible areas is $(0,ab]$.

1829. (a) The shaded area is

$$\int_0^1 1-x^2\,dx=\Big[x-\tfrac{1}{3}x^3\Big]_0^1=\tfrac{2}{3}.$$

Hence, we need to solve for k in the following:

$$\begin{aligned}&\int_0^k 1-x^2\,dx=\tfrac{1}{3}\\ \implies\ &\Big[x-\tfrac{1}{3}x^3\Big]_0^k=\tfrac{1}{3}\\ \implies\ &k-\tfrac{1}{3}k^3=\tfrac{1}{3}\\ \implies\ &k^3-3k+1=0,\text{ as required.}\end{aligned}$$

   (b) Since the root we are looking for has $|k|<1$, the most sensible rearrangement $k=\mathrm{g}(k)$ of the equation is $k=\frac{1}{3}(k^3+1)$: the cube will keep the numbers small, giving convergence. The iteration is $k_{n+1}=\frac{1}{3}(k_n^3+1)$. Starting with $k_0=0.5$, we get $k_n\to 0.34729....$ So, $k=0.347$ (3dp).

1830. Out of 64 outcomes in the possibility space, only HTHTHT and THTHTH are successful. Hence, the probability is $\frac{1}{32}$.

1831. (a) The vertices of the parabolae are at $(2,8)$ and $(7,8)$. So, they reach the same vertical height. Hence, they must have had the same initial vertical velocity, and so will remain at the same height as each other.
   (b) It is not guaranteed, because we don't know which direction each particle is travelling in. When one particle is at $({}^{11}/_3,{}^{22}/_9)$, the other particle could be at the same height but on the other side of its parabolic trajectory.

1832. According to the reverse chain rule, we multiply by the reciprocal of the derivative of the linear inside function:

$$\int e^{2x+1}\,dx=\tfrac{1}{2}e^{2x+1}+c.$$

— Nota Bene —

As ever, if you want to understand an integral such as the result above, simply carry out the relevant differentiation afterwards.

1833. (a) Firstly, we differentiate, giving $\frac{dy}{dx}=2x-1$. At $x=p$, this is $m=2p-1$. The tangent line passes through (p,p^2-p), so it has equation $y-(p^2-p)=(2p-1)(x-p)$, which we can simplify to $y=(2p-1)x-p^2$.
   (b) The point $(6,14)$ must satisfy this equation, so

$$\begin{aligned}&14=6(2p-1)-p^2\\ \implies\ &p=2,10.\end{aligned}$$

Hence, the tangents have equations $y=3x-4$ and $y=19x-100$.

1834. We require $(2z+1)^2$ to match the term in z^2. This provides $4z$, which means we need $3(2z+1)$ for the term in z. The constant term is now $1+3=4$, so we need $+15$. This gives

$$4z^2+10z+19\equiv(2z+1)^2+3(2z+1)+15.$$

1835. Using log rules, the curves are $y=\ln a+\ln x$ and $y=\ln b+\ln x$. These as both translations, parallel to the y axis, of the curve $y=\ln x$. Hence, they are translations of each other. The relevant vector is $\binom{0}{\ln b-\ln a}$ or $\ln\left(\frac{b}{a}\right)\mathbf{j}$.

1836. Using the chain rule,

$$\begin{aligned}&\frac{d}{dx}(1+y)^2+\frac{d}{dx}(1-y)^2\\ \equiv\ &2(1+y)\cdot\frac{dy}{dx}+2(1-y)\cdot-\frac{dy}{dx}\\ \equiv\ &4y\frac{dy}{dx}.\end{aligned}$$

1837. (a) We can use the fact that the sum of the first n integers is $\frac{1}{2}n(n+1)$. Proceeding algebraically,

$$\sum_0^{20}4n=4\sum_0^{25}n=4\times\tfrac{1}{2}n(n+1)\Big|_{n=25}$$

(b) Subtracting from the sum of all 100 integers,

$$\begin{aligned}S &= \tfrac{1}{2}n(n+1)\Big|_{n=100} - 4\times\tfrac{1}{2}n(n+1)\Big|_{n=25} \\ &= 5050 - 1300 \\ &= 3750.\end{aligned}$$

1838. Without this fact, $\mathbb{P}(10) = \frac{3}{36} = \frac{1}{12}$. With it, the possibility space is

$$\{(0,2),(1,3),(2,4),(3,5),(4,6), \text{ and vice versa}\}.$$

Of these ten outcomes, two give a total of 10. So $\mathbb{P}(10 \mid \text{this fact}) = \frac{2}{10} = \frac{1}{5}$. Since $\frac{1}{5} > \frac{1}{12}$, this fact increases the probability of a total of 10.

1839. Using $\log_{a^k} b^k \equiv \log_a b$, we can replace $\log_{e^3} x$ with $\log_e x^{\frac{1}{3}}$. This gives

$$\begin{aligned}&\ln x^{\frac{1}{3}} + \ln x = 4 \\ \Longrightarrow\ &\tfrac{1}{3}\ln x + \ln x = 4 \\ \Longrightarrow\ &\ln x = 3 \\ \Longrightarrow\ &x = e^3.\end{aligned}$$

1840. (a) Summing the roots, the $\pm\sqrt{\Delta}$ cancels, and we are left with $-b/a$.

(b) Multiplying the roots produces a difference of two squares. This simplifies to

$$\frac{b^2 - (b^2 - 4ac)}{4a^2} \equiv \frac{4ac}{4a^2} \equiv \frac{c}{a}.$$

1841. Two radii, length r, and one side form an isosceles triangle. Its central angle is $\frac{360°}{2n} = \frac{180°}{n}$. Splitting this triangle into two right-angled triangles, they have angle $\frac{90°}{n}$, opposite $\frac{1}{2}l$ and hypotenuse r. The diameter is given, then, by

$$d = 2r \equiv 2\frac{\frac{1}{2}l}{\sin\frac{90°}{n}} = l\operatorname{cosec}\frac{90°}{n}, \text{ as required.}$$

1842. It doesn't necessarily follow that the combined r is in the acceptance region.

A sample in the acceptance region means that there is *insufficient* evidence to reject H_0. But it might provide *some* evidence. It is possible that the evidence contained in the combined sample is sufficient for rejection of H_0, even if the evidence in each individual sample isn't.

1843. Assume, for a contradiction, that there is a quadratic factorisation $(3x^2+x+3)(ax^2+bx+c)$. Comparing coefficients:

$$\begin{aligned}&x^4: 27 = 3a \implies a = 9. \\ &x^3: 5 = a + 3b \implies b = -\tfrac{4}{3} \\ &x^2: 0 = 3a + b + 3c \implies c = -\tfrac{77}{9}.\end{aligned}$$

But this gives the constant term as $-\frac{77}{3} \neq -9$. Hence, such a factorisation is not possible.

1844. (a) $\dfrac{dx}{dt} = 2at^2 + b$.

(b) Substituting in, we require

$$\begin{aligned}&2at^2 + \frac{at^2+bt+c}{t} \equiv t^2 \\ \Longrightarrow\ &2at^3 + at^2 + bt + c \equiv t^3.\end{aligned}$$

The coefficient of t^2 requires $a = 0$, but this means the LHS has no term in t^3. Hence, there is no such solution to the differential equation.

1845. Using the formula for conditional probability,

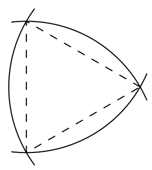
$$\begin{aligned}&\mathbb{P}(A \mid B) > \mathbb{P}(B \mid A) \\ \Longrightarrow\ &\frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)} > \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(A)} \\ \Longrightarrow\ &\frac{1}{\mathbb{P}(B)} > \frac{1}{\mathbb{P}(A)} \\ \Longrightarrow\ &\mathbb{P}(A) > \mathbb{P}(B).\end{aligned}$$

The direction of inequality is the same throughout because all probabilities must be positive.

1846. Using the sine area formula, the total area is $\frac{1}{2}\sin 2\theta$. Using $\frac{1}{2}bh$, each right-angled triangle has area $\frac{1}{2}\sin\theta\cos\theta$. Equating the expressions for the total area gives

$$\begin{aligned}&\tfrac{1}{2}\sin 2\theta \equiv \sin\theta\cos\theta \\ \Longrightarrow\ &\sin 2\theta \equiv 2\sin\theta\cos\theta, \text{ as required.}\end{aligned}$$

1847. (a) A Reuleaux triangle is composed of three arcs drawn as follows:



(b) The width of the shape is always, regardless of orientation, from a vertex to the opposite side. Hence, it always lies along the radius of one of the arcs. So, the width is r.

(c) Each of the segments has area given by sector minus triangle, which is

$$\tfrac{1}{2}r^2 \times \tfrac{\pi}{3} - \tfrac{1}{2}r \times \tfrac{\sqrt{3}}{2}r.$$

Adding three of these to the area of the central triangle, the total area is

$$\begin{aligned}&3\Big(\tfrac{1}{2}r^2\times\tfrac{\pi}{3} - \tfrac{1}{2}r\times\tfrac{\sqrt{3}}{2}r\Big) + \tfrac{1}{2}r\times\tfrac{\sqrt{3}}{2}r \\ \equiv\ &\tfrac{1}{2}\big(\pi - \sqrt{3}\big)r^2, \text{ as required.}\end{aligned}$$

1848. The individual sets are $I = (-4, 6)$ and $J = [1, \infty)$. The intersection, therefore, is $[1, 6)$.

1849. The implication is $\Longleftarrow$. If $x \in \{a, b\}$, then either $(x-a)$ or $(x-b)$ is zero, so $(x-a)(x-b)(x-c) = 0$. The counterexample to the forwards implication is $x = c$.

1850. The graph $y = \ln \frac{1}{2}x$ can be rewritten as $2e^y = x$. This is the same as $y = 2e^x$, with the roles of x and y reversed. Hence, the graphs are reflections in the line $y = x$.

1851. The shape must have one pair of parallel sides; these must be the sides with vectors $\mathbf{a}$ and $k\mathbf{a}$. And, since these vectors run tip-to-tail, $\mathbf{a}$ and $k\mathbf{a}$ must be antiparallel, i.e. k must be negative. However, this includes one possibility which would make the shape a parallelogram. Excluding this, $k \in (-\infty, 0) \setminus \{-1\}$.

1852. The RHS may be rewritten as follows:

$$\begin{aligned} & q^{\log_b a} \\ = & \left(b^x\right)^{\log_b a} \\ \equiv & \left(b^{\log_b a}\right)^x \\ \equiv & \, a^x \\ = & \, p, \text{ as required.} \end{aligned}$$

1853. Differentiating the first equation,

$$12\frac{dx}{dt} = 6t - 2t^{-3} - 4.$$

Substituting this and x into the second equation, the LHS is

$$\begin{aligned} & t\left(\tfrac{1}{2}t - \tfrac{1}{6}t^{-3} - \tfrac{1}{3}\right) + 2\left(\tfrac{1}{4}t^2 + \tfrac{1}{12}t^{-2} - \tfrac{1}{3}t + \tfrac{1}{2}\right) \\ \equiv & \tfrac{1}{2}t^2 - \tfrac{1}{6}t^{-2} - \tfrac{1}{3}t + \tfrac{1}{2}t^2 + \tfrac{1}{6}t^{-2} - \tfrac{2}{3}t + 1 \\ \equiv & \, t^2 - t + 1. \end{aligned}$$

This is the RHS of the second equation, which proves the required implication.

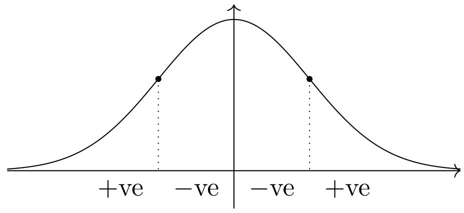
1854. There are two cases:

① If the central square is shaded, then its four neighbours cannot be. This leaves only the four corner squares, which must then all be shaded.

② If the central square is not shaded, then, since the remaining squares form a ring, it is not possible to shade more than every other square. This gives a maximum of four shaded squares in total.

Hence, overall, there is only one way of shading five squares, as in case ①. QED.

1855. (a) The trapezium rule over or underestimates area according to the curvature. For a bell curve, the sign of the second derivative is shown below:



In these case, therefore, it

i. underestimates: the curvature is negative.
ii. overestimates: the curvature is positive.

(b) Good approximations will occur at points where the normal distribution curve is close to being linear: where the second derivative is close to zero. The second derivative is zero at points of inflection. On a normal distribution, these are at $|X - \mu| = \sigma$, as marked on the diagram.

1856. The roots do match, with a double root at $x = 1$ and a single root at $x = -1$. However, the graph is a negative cubic, whereas the equation suggested has leading coefficient $+1$. So, the given equation could not be that of the given graph.

1857. Substituting for y, we get a quadratic in $\sqrt{x}$:

$$\begin{aligned} & 2\sqrt{x} + 2\sqrt{x^2} = 3 \\ \Longrightarrow \; & 2x + 2\sqrt{x} - 3 = 0 \\ \Longrightarrow \; & \sqrt{x} = \frac{-2 \pm \sqrt{28}}{4} \end{aligned}$$

We take the positive root, as $\sqrt{x} \geq 0$, which gives

$$x = \tfrac{1}{2}(4 - \sqrt{7}), \quad y = \tfrac{1}{4}(23 - 8\sqrt{7}).$$

1858. The translations have no effect on area, but the stretches do. We have a stretch parallel to the y axis by scale factor a and a stretch parallel to the x axis by scale factor $1/b$. Hence, overall, the area scale factor is a/b.

1859. Using $\cos(x \pm y) \equiv \cos x \cos y \mp \sin x \sin y$,

$$\begin{aligned} & \frac{dy}{dx} = 2\sin x \sin y \\ \Longrightarrow \; & \operatorname{cosec} y \frac{dy}{dx} = 2\sin x. \end{aligned}$$

1860. We can set aside the scale factor l, reintroducing it as an area scale factor l^2 at the end. The lengths in GP are $1, \frac{3}{2}, \frac{9}{4}$. Using the cosine rule, the angle between the two longer sides is

$$\cos\theta = \frac{\frac{3}{2}^2 + \frac{9}{4}^2 - 1^2}{2 \times \frac{3}{2} \times \frac{9}{4}} = \frac{101}{108}.$$

Hence, using $\sin^2\theta + \cos^2\theta \equiv 1$,

$$\sin\theta = \sqrt{1 - \frac{101^2}{108^2}} = \frac{\sqrt{1463}}{108}.$$

The sine area formula gives

$$A_\triangle = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{9}{4} \cdot \frac{\sqrt{1463}}{108}.$$

Reinstating the area scale factor,

$$A_\triangle = \frac{\sqrt{1463}}{64} l^2.$$

1861. A successful outcome is where

$$\begin{aligned} & X^2 - 1 > X \\ \Longrightarrow\ & X^2 - X - 1 > 0 \\ \Longrightarrow\ & X < \tfrac{1-\sqrt{5}}{2} \text{ or } X > \tfrac{1+\sqrt{5}}{2}. \end{aligned}$$

Using a cumulative distribution function,

$$\begin{aligned} \mathbb{P}(X^2 - 1 > X) &= 1 - P\left(\tfrac{1-\sqrt{5}}{2} < X < \tfrac{1+\sqrt{5}}{2}\right) \\ &= 1 - 0.678896... \\ &= 0.321 \text{ (3sf)}. \end{aligned}$$

1862. (a) We first write the base as 4^k, then use index laws to manipulate the expression:

$$\begin{aligned} 2^{6-2x} &\equiv \left(4^{\frac{1}{2}}\right)^{6+2x} \\ &\equiv 4^{3+x} \\ &\equiv 4^3 \times 4^x \\ &\equiv 64 \times 4^x. \end{aligned}$$

(b) This time, we use $4^{\frac{3}{2}} = 8$:

$$\begin{aligned} 8^{4-2x} &\equiv \left(4^{\frac{3}{2}}\right)^{4-2x} \\ &\equiv 4^{6-3x} \\ &\equiv \frac{4^6}{4^{3x}} \\ &\equiv \frac{4096}{(4^x)^3}. \end{aligned}$$

1863. (a) $\sqrt{3} \star \sqrt{27} = 3 + \sqrt{3}\sqrt{27} + 27 = 39$.

(b) This yields a quadratic:

$$\begin{aligned} & x \star (x+2) = 1 \\ \Longrightarrow\ & x^2 + x(x+2) + (x+2)^2 = 1 \\ \Longrightarrow\ & 3x^2 + 6x + 3 = 0 \\ \Longrightarrow\ & (x+1)^2 = 0 \\ \Longrightarrow\ & x = -1. \end{aligned}$$

1864. (a) The second term x_1 is

$$x_1 = \frac{1}{a + 1 + b\sqrt{2}}.$$

Substituting this, we get an inlaid fraction:

$$\begin{aligned} x_2 &= \frac{1}{\frac{1}{a+1+b\sqrt{2}} + 1} \\ &\equiv \frac{a + 1 + b\sqrt{2}}{a + 2 + b\sqrt{2}}. \end{aligned}$$

(b) The same again gives

$$\begin{aligned} x_3 &= \frac{1}{\frac{a+1+b\sqrt{2}}{a+2+b\sqrt{2}} + 1} \\ &\equiv \frac{a + 2 + b\sqrt{2}}{a + 3 + b\sqrt{2}}. \end{aligned}$$

Spotting the pattern, the ordinal formula is

$$x_n = \frac{a + n - 1 + b\sqrt{2}}{a + n + b\sqrt{2}}.$$

Splitting the fraction up,

$$x_n = 1 - \frac{1}{a + n + b\sqrt{2}}.$$

1865. (a) False. The critical region and the acceptance region are mutually exclusive and exhaustive, i.e. they are complementary. In less formal terms, they are opposites. Every sample must lie in one or the other.

(b) True. The statements "the sample has a test statistic lying in the critical region" and "the sample's p-value is less than the significance level" are two ways of saying exactly the same thing, in the languages of test statistics or probabilities.

1866. There are ${}^8\text{C}_3$ ways of selecting three vertices from the eight. There are 8 locations in which to place a set of three adjacent vertices, so there are 8 successful outcomes. Hence, the probability is $8/{}^8\text{C}_3 = \frac{1}{7}$.

— Alternative Method —

The octagon is symmetrical, so we can choose the first vertex arbitrarily without loss of generality. There are now ${}^7\text{C}_2 = 21$ equally likely ways of choosing the other two vertices. Of these, three (both one side, both the other, one either side) are successful. So, the probability is $\frac{3}{21} = \frac{1}{7}$.

1867. The standard equation for fixed points is $x = \text{f}(x)$. So, we require that $x = x^2 + px + q$ has exactly one root. Rearranging to $x^2 + (p-1)x + q = 0$, we set the discriminant to zero:

$$\begin{aligned} & (p-1)^2 - 4q = 0 \\ \Longrightarrow\ & p = 1 \pm 2\sqrt{q}. \end{aligned}$$

1868. Assume, for a contradiction, that a frequency ratio $k : 1$, where $k > 1$, can be expressed as a whole number of octaves and also as a whole number of fifths. This means that $k = 2^p$ for $p \in \mathbb{N}$ and $k = \frac{3}{2}^q$, where $q \in \mathbb{N}$. Equating these,

$$\begin{aligned} 2^p &= \tfrac{3}{2}^q \\ \Longrightarrow\ 2^{p+q} &= 3^q. \end{aligned}$$

But the only possible prime factors on the LHS and RHS are 2 and 3 respectively. Hence, this equation can only hold when $p = q = 0$, so $k = 1$. But $k > 1$. This is a contradiction. So, no interval can be expressed both as a whole number of octaves and as a whole number of fifths. □

1869. The intersections are where

$$\begin{aligned} & x^2 = 1 + x - x^2 \\ \Longrightarrow\ & 2x^2 - x - 1 = 0 \\ \Longrightarrow\ & (2x+1)(x-1) = 0 \\ \Longrightarrow\ & x = -\tfrac{1}{2}, 1. \end{aligned}$$

The negative parabola is above the positive one between these intersections, so the area is

$$\begin{aligned} & \int_{-\frac{1}{2}}^{1} 1 + x - 2x^2 \, dx \\ &= \left[x + \tfrac{1}{2}x^2 - \tfrac{2}{3}x^3\right]_{-\frac{1}{2}}^{1} \\ &= \left(1 + \tfrac{1}{2} - \tfrac{2}{3}\right) - \left(-\tfrac{1}{2} + \tfrac{1}{8} + \tfrac{2}{24}\right) \\ &= \tfrac{9}{8}. \end{aligned}$$

1870. Both of these results follow from the same result: the definitions of sin and cos on the unit circle are symmetrical in $y = x$, or equivalently in $\theta = 45°$.

(a) $(90° - \theta)$ and θ are symmetrical in $\theta = 45°$.

(b) $(45° + \theta)$ and $(45° - \theta)$ have the same symmetry.

1871. We propose a root $p + q\sqrt{2}$, for $p, q \in \mathbb{N}$. So,

$$\begin{aligned} & \left(p + q\sqrt{2}\right)^2 = 33 + 8\sqrt{2} \\ \Longrightarrow\ & p^2 + 2q^2 + 2pq\sqrt{2} = 33 + 8\sqrt{2}. \end{aligned}$$

Since $p, q \in \mathbb{N}$, we can equate coefficients, giving

$$\begin{aligned} p^2 + 2q^2 &= 33, \\ 2pq &= 8. \end{aligned}$$

Substituting the latter into the former,

$$\begin{aligned} & p^2 + \frac{32}{p^2} = 33 \\ \Longrightarrow\ & p^4 - 33p^2 + 32 = 0 \\ \Longrightarrow\ & (p^2 - 32)(p^2 - 1) = 0 \\ \Longrightarrow\ & p = \pm\sqrt{32}, \pm 1. \end{aligned}$$

Since $p \in \mathbb{N}$, we need $p = 1$. Hence, $q = 4$. This gives the required square root as $1 + 4\sqrt{2}$.

1872. Using the product rule,

$$\begin{aligned} \frac{d}{dt}(xy) &= \frac{dx}{dt}y + x\frac{dy}{dt} \\ &= ay + bx. \end{aligned}$$

1873. Differentiating,

$$\begin{aligned} & y = x^{\frac{1}{3}} \\ \Longrightarrow\ & \frac{dy}{dx} = \tfrac{1}{3}x^{-\frac{2}{3}} \\ \Longrightarrow\ & \frac{d^2y}{dx^2} = -\tfrac{2}{9}x^{-\frac{5}{3}}. \end{aligned}$$

This is undefined as $0^{-\frac{5}{2}}$ involves division by zero. However, $y = \sqrt[3]{x}$ is a reflection of $y = x^3$ in the line $y = x$, and $y = x^3$ is inflected at the origin. Hence, so is $y = \sqrt[3]{x}$.

Nota Bene

The conceptual point here is: the curve $y = \sqrt[3]{x}$ is inflected at the origin, but it is not possible to *show* that it is so using derivatives with respect to x, because they are undefined at $x = 0$. Derivatives with respect to y would, however, do the trick, as $y = \sqrt[3]{x}$ is the same curve as $x = y^3$.

1874. (a) This is not true. Both $\text{f}(x) = 10x^2 + 1$ and $\text{g}(x) = x^2 + 2$ have the same S, viz. the empty set, yet their graphs intersect.

(b) This is true. If S is not empty, then any value x for which $\text{f}(x) = 0$ and $\text{g}(x) = 0$ must satisfy $\text{f}(x) = \text{g}(x)$. Hence, anything in S is in the solution set of E. And if S is empty, then it is automatically a subset.

(c) This is not true, for the same reason as in (a).

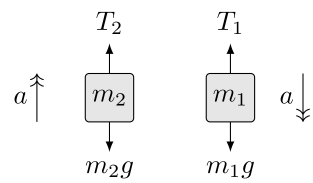
1875. (a) Using a log law, the sequence is

$$\ln k, 2\ln k, 3\ln k, ..., 9\ln k$$

Hence, it is an AP with first term $\ln k$ and last term $9\ln k$.

(b) The mean of an AP is the mean of first and last, which is $5\ln k$. This is also the median.

1876. The string is inextensible, so, despite the friction in the pulley, the accelerations will be the same. The tensions, however, will not:



At one extreme, without any friction in the pulley, we would have $T_1 = T_2$. At the other, with very high friction, i.e. the pulley so stiff as to maintain equilibrium, we would have

$$T_1 = m_1 g > m_2 g = T_2.$$

This scenario is between the two, so $T_1 > T_2$. And, since the system accelerates, we know that $T_1 < m_1 g$ and $T_2 > m_2 g$. Hence, we have

$$m_2 g < T_2 < T_1 < m_1 g.$$

1877. In each case, the domain is $\mathbb{R}$, excluding any real numbers which make the denominator zero.

(a) $\mathbb{R} \setminus \{-1\}$,

(b) $\mathbb{R}$,

(c) $\mathbb{R} \setminus \{-1\}$.

1878. We can use the factor theorem. For the algebraic fraction to simplify to a polynomial, $(3x+5)$ would have to be a factor of the numerator. But, when we substitute $x = -\frac{5}{3}$, we get

$$36\left(-\tfrac{5}{3}\right)^3 + 216\left(-\tfrac{5}{3}\right)^2 + 233\left(-\tfrac{5}{3}\right) - 48 = -3.$$

Since this is non-zero, the algebraic fraction can't be simplified to a polynomial.

1879. We need x_1, x_2 such that $\mathbb{P}(X < x_1) = 0.1$ and $\mathbb{P}(X < x_2) = 0.9$. The inverse normal facility on a calculator gives, to 3sf, $x_1 = 36.2$ and $x_2 = 43.8$.

1880. Writing the sum longhand, we can integrate:

$$\begin{aligned}&\int_1^2 \sum_{i=1}^{3} x^{-i}\, dx \\ &= \int_1^2 x^{-1} + x^{-2} + x^{-3}\, dx \\ &= \Big[\ln|x| - x^{-1} - \tfrac{1}{2}x^{-2}\Big]_1^2 \\ &= \left(\ln 2 - \tfrac{1}{2} - \tfrac{1}{8}\right) - \left(\ln 1 - 1 - \tfrac{1}{2}\right) \\ &= \tfrac{7}{8} + \ln 2, \text{ as required.}\end{aligned}$$

1881. Giving the outer square side length 1, the outer set of shaded triangles has area $\frac{1}{2}$. The length scale factor to the first inscribed square is $1/\sqrt{2}$, so the LSF to the second inscribed square is $1/2$. Hence, the area scale factor to the second set of shaded triangles is $\text{LSF}^2 = 1/4$. So, the total shaded area is an infinite geometric series with first term $a = 1/2$ and common ratio $r = 1/4$. Using the standard sum:

$$S_\infty = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{2}{3}.$$

The ratio of areas is, therefore, $2 : 1$.

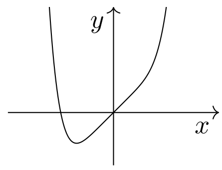
1882. We want no acceleration in the formula. So, we can rearrange the second equation to the form $a = ...$ and equate:

$$\begin{aligned}& s = ut + \tfrac{1}{2}\frac{v^2 - u^2}{2s}t^2 \\ \Longrightarrow\ & s^2 = sut + \tfrac{1}{4}(v^2 - u^2)t^2 \\ \Longrightarrow\ & 4s^2 - 4sut + u^2t^2 - v^2t^2 = 0 \\ \Longrightarrow\ & 4s^2 - 4sut + u^2t^2 = v^2t^2.\end{aligned}$$

Factorising the LHS,

$$\begin{aligned}& (2s - ut)^2 = v^2t^2 \\ \Longrightarrow\ & 2s - ut = \pm vt \\ \Longrightarrow\ & 2s = (u \pm v)t \\ \Longrightarrow\ & s = \tfrac{1}{2}(u \pm v)t.\end{aligned}$$

1883. For a point of inflection, it is not sufficient that the second derivative be zero; it must change sign. Here, the second derivative is $\frac{d^2y}{dx^2} = 30x^4$. This is zero at the origin but positive either side of it. Since the second derivative does not change sign, this is not a point of inflection.



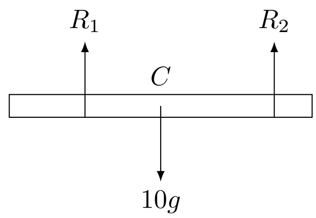
1884. (a) $A \setminus B$,

(b) $\varnothing$,

(c) B.

1885. This is true. In a lower-dimensional analogy, one linear equation in two unknowns can only produce a line, never a unique value. The same holds true in 3D: each equation restricts the solution space by one dimension, so 3D becomes 2D becomes 1D. The solution to a set of two linear equations in three unknowns must be a line or nothing, never a unique point.

1886. (a) The force diagram is

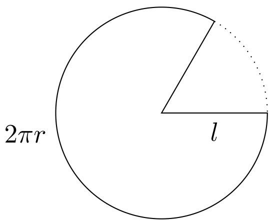


$$\begin{aligned}\updownarrow &: R_1 + R_2 - 10g = 0 \\ \overset{\frown}{C} &: R_2 \times 1.5 - R_1 \times 1 = 0\end{aligned}$$

So, $R_1 = 6g$ and $R_2 = 4g$.

(b) Because the velocity is constant, the fact that the workmen are on a slope doesn't change anything. The acceleration is still zero, and the ratio between the distances from C is as before. So, $R_1 = 6g$ and $R_2 = 4g$ again.

1887. The unwrapped sector is as follows:



Of a total circumference (including the dotted arc above) $2\pi l$, we have $2\pi r$, so the sector above is $\frac{r}{l}$ of the circle. Hence, the area is

$$\frac{r}{l} \times \pi l^2 \equiv \pi r l.$$

Combined with the base of the cone, which has area πr^2, the area formula is $A = \pi r(r + l)$. $\square$

1888. (a) This is not well defined. The inside function $1 + \mathrm{g}(x)$ has range $[1, 2]$, which is outside the domain of g.

(b) This is well defined. The range of $1 - \mathrm{g}(x)$ is $[0, 1]$, which is precisely the domain of g.

1889. The standard partial sum formula is

$$S_n = \tfrac{1}{2}n\big(2a + (n-1)d\big).$$

Using this,

$$\begin{aligned} & 29385 = \tfrac{1}{2}n\big(30 + 7(n-1)\big) \\ \Longrightarrow\; & 7n^2 + 23n - 58770 = 0 \\ \Longrightarrow\; & n = -\tfrac{653}{7}, 90. \end{aligned}$$

Since this is an arithmetic series with a first term at $n = 1$, we want $n = 90$.

1890. (a) The intersections satisfy a cubic equation. So, there could be $\{1, 2, 3\}$ intersections.

(b) Generally, the equation for intersections is a cubic. But, if the leading coefficients match, then the equation for intersections has degree less than 3. For example, $y = x^3$ and $y = x^3+1$ do not intersect at all. So, the possibilities are $\{0, 1, 2, 3\}$.

(c) The intersections satisfy a quartic. Hence, any number of roots $\{0, 1, 2, 3, 4\}$ is possible.

1891. Scaling the sample size by 4 scales the standard deviation of sample means by $\frac{1}{2}$. So, the standard deviation is 0.06.

1892. (a) Generalising the unit circle to any radius, $(x, y) = (r\cos\theta, r\sin\theta)$.

(b) Dividing, $\tan\theta = \frac{y}{x}$ and $\cot\theta = \frac{x}{y}$.

(c) Dividing both sides by $\sin\theta$,

$$\begin{aligned} & r\sin^2\theta = \cos\theta \\ \Longrightarrow\; & r\sin\theta = \cot\theta \\ \Longrightarrow\; & y = \tfrac{x}{y} \\ \Longrightarrow\; & y^2 = x. \end{aligned}$$

This is a parabola, as required.

1893. Integrating the second derivative $\mathrm{f}''(x)$,

$$\mathrm{f}'(x) = 2x^2 + c.$$

Substituting $\mathrm{f}'(2) = 6$ gives $6 = 8 + c$, so $c = -2$. Therefore, the first derivative is

$$\mathrm{f}'(x) = 2x^2 - 2.$$

For stationary points,

$$\begin{aligned} & 2x^2 - 2 = 0 \\ \Longrightarrow\; & x = \pm\sqrt{2}. \end{aligned}$$

1894. Rearranging,

$$\begin{aligned} & 3\sin^2 x - \cos^2 x = 0 \\ \Longrightarrow\; & 3\sin^2 x = \cos^2 x \\ \Longrightarrow\; & \tan x = \pm\tfrac{1}{\sqrt{3}}. \end{aligned}$$

The primary values are $x = \pm 30°$. Other values are generated by adding multiples of 180° to the primary values. Hence, the equation has solution set $\{x : x = 180°n \pm 30°, n \in \mathbb{Z}\}$, as required.

1895. The minimum speed will be attained at 45° above the horizontal. With initial speed u, this gives components of $\frac{\sqrt{2}}{2}u$. The *suvat*s are

$$\begin{aligned} 0 &= \tfrac{\sqrt{2}}{2}ut - \tfrac{1}{2}gt^2, \\ d &= \tfrac{\sqrt{2}}{2}ut. \end{aligned}$$

Rearranging the latter for t, $t = \frac{d\sqrt{2}}{u}$. Substituting this into the former,

$$0 = d - \frac{d^2 g}{u^2}.$$

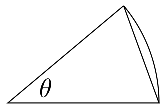
Solving this, $u_{\min} = \sqrt{dg}$ ms^{-1}.

1896. Multiplying up by the denominators,

$$1 \equiv P(x^2 + 1) + Qx \equiv Px^2 + Qx + P.$$

Equating constant terms, $P = 1$. But coefficients of x^2 then give $0 = 1$. So, such constants cannot be found.

1897. With θ exaggerated for clarity, the scenario is



The sector has area $\frac{1}{2}r^2\theta$. The triangle has area $\frac{1}{2}r^2\sin\theta$. Using the cubic approximation to the sine function, this gives the area of the segment as

$$\begin{aligned} A_{\text{seg}} &= \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2\sin\theta \\ &\equiv \tfrac{1}{2}r^2(\theta - \sin\theta) \\ &\approx \tfrac{1}{2}r^2\Big(\theta - \big(\theta - \tfrac{1}{6}\theta^3\big)\Big) \\ &\equiv \tfrac{1}{12}r^2\theta^3, \text{ as required.} \end{aligned}$$

1898. Writing 2 over base e,

$$\begin{aligned} \frac{d}{dx}2^x &\equiv \frac{d}{dx}\left(e^{\ln 2}\right)^x \\ &\equiv \frac{d}{dx}e^{x\ln 2} \\ &\equiv \ln 2 \cdot e^{x\ln 2} \\ &\equiv \ln 2 \cdot 2^x, \text{ as required.} \end{aligned}$$

1899. Since the x_i values to be increased are chosen at random, we can take a weighted average of the scale factors, noting that the variance is scaled by the square of the coding scale factor. Hence, the calculation is s.f. $= 1 \times 0.8 + 1.25 \times 0.2 = 1.05$. The squared scale factor is $1.05^2 = 1.1025$, so the variance scales, on average, by 10.25%.

1900. Multiplying up and using a log rule,

$$\begin{aligned} & \frac{2\ln x}{\ln(5x-4)} = 1 \\ \Longrightarrow\ & 2\ln x = \ln(5x-4) \\ \Longrightarrow\ & \ln x^2 = \ln(5x-4) \end{aligned}$$

We can now exponentiate:

$$\begin{aligned} & x^2 = 5x - 4 \\ \Longrightarrow\ & x = 1, 4. \end{aligned}$$

However, the root $x = 1$ gives division by $\ln 1 = 0$ in the initial equation. So, the solution is $x = 4$.

——— End of 19th Hundred ———